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On Monosplines with Nonnegative Coefficients

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In this paper we establish some inequalities for monosplines and apply them to best quadrature formulas for certain classes of functions with a nonsymmetric norm.

1

Let $w(t)$ be an integrable function on $[0, 1]$ such that

$$\text{meas } E(w \leqslant 0) = 0, \tag{1}$$

let $r \geqslant 1$ be an integer and let A, B be given subsets (possibly empty) of $Z_r = \{0, ..., r-1\}$.

$M_N^r(A, B)$ denotes the set of monosplines

$$\begin{gathered} M(x) = w_r(x) - \sum_{i=1}^{n} \sum_{j=0}^{r-1} a_{ij}(x - x_i)_+^{r-1-j} + \sum_{k=0}^{r-1} b_k x^k, \\ \sum_{i=1}^{n} \sum_{j=0}^{r-1} \operatorname{sgn} |a_{ij}| \leqslant N \end{gathered} \tag{2}$$

which satisfy the boundary conditions

$$M^{(i)}(0) = 0 \quad (i \in A), \qquad M^{(j)}(1) = 0 \quad (j \in B), \tag{3}$$

where

$$w_r(x) = \int_0^1 w(t)(x-t)_+^{r-1}\, dt, \qquad u_+^m = \begin{cases} u^m, & u > 0 \\ 0, & u \leqslant 0. \end{cases}$$

Also, let

$$M_N^{r0}(A, B) := M_N^r(A, B) \cap C^{r-2}[0, 1].$$

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The monosplines $M \in M_N^{r0}(A, B)$ have the form

$$M(x) = w_r(x) - \sum_{i=1}^{N} a_i (x - x_i)_+^{r-1} + \sum_{k=0}^{r-1} b_k x^k. \tag{4}$$

Finally, let $M_N^{r+}(A, B)$ denote the set of all monosplines $M \in M_N^{r0}(A, B)$ which have nonnegative coefficients a_i $(i = 1 : N)$ in the representation (4).

Let M_N^r, M_N^{r0}, M_N^{r+} be the corresponding sets of 1-periodic monosplines. They have the representation

$$M(x) = w_r(x) - \sum_{i=1}^{n} \sum_{j=1}^{r-1} a_{ij} D_{r-j}(x - x_i) + a_0, \qquad \sum_{i=1}^{n} a_{i0} = \int_0^1 w(t)\, dt,$$
$$\sum_{i=1}^{n} \sum_{j=0}^{r-1} \operatorname{sgn} |a_{ij}| \leqslant N, \tag{5}$$

where $x_1 < \cdots < x_n < x_1 + 1$,

$$w_r(x) = \int_0^1 w(t)\, D_r(x - t)\, dt,$$

$$D_m(u) = (m-1)!/(2^{m-1}\pi^m) \sum_{k=1}^{\infty} k^{-m} \cos(2\pi k u - \pi m/2)$$

(in this case $w(t)$ is a 1-periodic function). If $M \in M_N^{r0}$ then

$$M(x) = w_r(x) - \sum_{i=1}^{N} a_i D_r(x - x_i) + a_0, \qquad \sum_{i=1}^{N} a_i = \int_0^1 w(t)\, dt. \tag{6}$$

The monosplines $M \in M_N^{r+}$ have nonnegative coefficients a_i $(i = 1 : N)$ in representation (6). We deduce from (2) and (5) that

$$M^{(r)}(x) = (r-1)!\, w(x) \text{ almost everywhere on } [0, 1], \tag{7}$$

$$a_{ij} = (M^{(r-1-j)}(x_i - 0) - M^{(r-1-j)}(x_i + 0))/(r-1-j)! \qquad (i = 1 : n;\, j = 0 : r-1). \tag{8}$$

In view of (3), (7), (1) we have

$$\nu(M) \leqslant 2N + r - |A| - |B| =: \nu \qquad \forall M \in M_N^r(A, B),$$
$$\nu(M) \leqslant 2N \qquad \forall M \in M_N^r,$$

where $\nu(f)$ is the number of zeros of f on (0, 1) (or on the period) counting multiplicities (see, e.g., [1]), and $|G|$ is the number of elements of set G. If $M \in M_N^{r0}(A, B)$ $(M \in M_N^{r0})$ satisfies $\nu(M) = \nu$ $(\nu(M) = 2N)$ then in view of (8) $M \in M_N^{r+}(A, B)$ $(M \in M_N^{r+})$.

By $\mu(f)$ we denote the number of sign changes of f on $[0, 1]$ (or on the period). For monosplines we have

$$\mu(M) \leqslant v \qquad \forall M \in M_N^r(A, B), \qquad \mu(M) \leqslant 2N \quad \forall M \in M_N^r.$$

LEMMA. *Let $U(x)$ and $V(x)$ be two splines*

$$U(x) = \int_0^x u(t)\, dt - \sum_{i=1}^m a_i (x - x_i)_+^0 + a_0,$$

$$V(x) = \int_0^x v(t)\, dt - \sum_{i=1}^n b_i (x - y_i)_+^0 + b_0,$$

where u and v are an integrable on $[0, 1]$ functions and

$$\text{meas } E(u < v) = 0. \tag{9}$$

Then the difference $s(x) = U(x) - V(x)$ has at most $2n_i + 1$ sign changes on (x_{i-1}, x_i) $(i = 1 : m+1;\ x_0 = 0,\ x_{m+1} = 1)$, where n_i is the number of points $y_j \in (x_{i-1}, x_i)$ such that the corresponding coefficients b_j are negative $(0 \leqslant n_i \leqslant n)$. If $s(x)$ has $2n_i + 1$ sign changes on (x_{i-1}, x_i) then $s(x_{i-1} + 0) < 0$, $s(x_i - 0) > 0$.

Proof. In view of (9) the difference $s(x)$ increases on each interval which does not contain the points $x_1, ..., x_m, y_1, ..., y_n$. Hence, $s(x)$ can change sign on this interval from "minus" to "plus." At the points $y_j \in (x_{i-1}, x_i)$ for which the corresponding coefficients b_j are negative the function $s(x)$ can change the sign from "plus" to "minus" also, because

$$s(y_j - 0) - s(y_j + 0) = b_j \geqslant 0.$$

Thus, $s(x)$ can change sign from "plus" to "minus" on (x_{i-1}, x_i) at most n_i times and the lemma is proved. ∎

COROLLARY. *Let $M_0 \in M_N^{r+}(A, B)$ and $c \in [0, 1]$ be fixed. Then for every $M \in M_N^r(A, B)$*

$$\mu(M^{(r-1)} - cM_0^{(r-1)}) \leqslant 2N + 1 - A_{r-1} - B_{r-1}, \tag{10}$$

where

$$A_{r-1} = \begin{cases} 1, & r-1 \in A \\ 0, & r-1 \notin A, \end{cases} \qquad B_{r-1} = \begin{cases} 1, & r-1 \in B \\ 0, & r-1 \notin B. \end{cases}$$

For $M_0 \in M_N^{r+}$ and $M \in M_N^r$ we have

$$\mu(M^{(r-1)} - cM_0^{(r-1)}) \leqslant 2N. \tag{11}$$

2

THEOREM 1. *Let* $M \in M_N^{r0}(A, B)$ *and* $\nu(M) = \nu$. *Then*

$$\begin{aligned} |M^{(k)}(0)| &\leqslant |M_0^{(k)}(0)| \; \|M\|/\|M_0\| \\ |M^{(k)}(1)| &\leqslant |M_0^{(k)}(1)| \; \|M\|/\|M_0\| \qquad (k = 0 : r-1), \end{aligned} \tag{12}$$

where M_0 *is the monospline of minimal* L_∞-*norm in* $M_N^r(A, B)$ ($M_0 \in M_N^{r+}(A, B)$, *see, e.g.*, [1]), $\|\cdot\| = \|\cdot\|_\infty$.

Proof. Since $\nu(M) = \nu$, $M \in M_N^{r+}(A, B)$ and

$$\operatorname{sgn} M^{(k)}(0) = \operatorname{sgn} M_0^{(k)}(0), \qquad \operatorname{sgn} M^{(k)}(1) = \operatorname{sgn} M_0^{(k)}(1) \qquad (k = 0 : r-1).$$

If $|M^{(k)}(0)| \leqslant |M_0^{(k)}(0)|$ then the inequality (12) holds, because

$$\|M_0\| \leqslant \|M\| \qquad \forall M \in M_N^r(A, B).$$

Assume that there exists a monospline $M \in M_N^{r0}(A, B)$ such that $\nu(M) = \nu$ and for fixed k $|M^{(k)}(0)| > |M_0^{(k)}(0)|$,

$$|M^{(k)}(0)| > |M_0^{(k)}(0)| \; \|M\|/\|M_0\|.$$

The monospline M_0 has $\nu + 1$ alternation points $0 \leqslant z_1 < \cdots < z_{\nu+1} \leqslant 1$ (see [1]).

$$|M_0(z_i)| = \|M_0\|, \qquad M(z_i) \cdot M(z_{i+1}) < 0.$$

Hence the difference

$$s(x) = M_0(x) - c_k M(x), \qquad c_k = M_0^{(k)}(0)/M^{(k)}(0), \qquad c_k \in (0, 1)$$

has ν sign changes on $[0, 1]$: $\mu(s) \geqslant \nu$. Thus,

$$\mu(s^{(k)}) \geqslant \nu - k + \alpha_k + \beta_k, \tag{13}$$

where $\alpha_k(\beta_k)$ is the number of elements of $A(B)$ which are less than k. Since $s^{(k)}(0) = 0$ we have,

$$\mu(s^{(r-1)}) \geqslant 2N + 2 - A_{r-1} - B_{r-1}, \qquad k < r-1.$$

This inequality contradicts (10). If $k = r-1$ then $A_{r-1} = 0$ and by the lemma $\mu(s^{(r-1)}) \leqslant 2N - B_{r-1}$. This inequality contradicts (13). Theorem 1 is proved. ∎

The following result is valid for a periodic setting.

THEOREM 2. *Let $w(t) \equiv \text{const} \neq 0$, then for every $M \in M_N^{r+}$*

$$\|M^{(k)}\| \leqslant \|M\| \ \|M_0^{(k)}\|/\|M_0\| \qquad (k=0:r-1), \tag{14}$$

where M_0 is the monospline with minimal L_∞-norm in M_N^r ($M_0 \in M_N^{r+}$, see, e.g., [1, 2]).

Remark. The monospline M_0 has the form

$$M_0(x) = N^{-r}(-D_r(Nx) + c_r),$$

where c_r is the constant of the best uniform approximation of $D_r(x)$,

$$\|D_r - c_r\| = \inf \|D_r - c\| =: K_r.$$

The inequality (14) can be rewritten in the form

$$\|M^{(k)}\| \leqslant N^k \ \|D_{r-k}\| \cdot \|M\|/K_r \qquad (k=1:r-1).$$

Proof. The monospline M_0 has $2N$ alternation points $z_1 < \cdots < z_{2N} < z_1 + 1$. If $\|M^{(k)}\| < \|M_0^{(k)}\|$ $(1 \leqslant k \leqslant r-1)$ then the inequality (14) holds because $\|M_0\| < \|M\|$.

Let $M(x)$ be a monospline in M_N^{r+} such that for fixed k $(1 \leqslant k \leqslant r-1)$

$$\|M^{(k)}\| > \|M_0^{(k)}\| \tag{15}$$

and

$$\|M^{(k)}\| > \|M_0^{(k)}\| \cdot \|M\|/\|M_0\|. \tag{16}$$

Let z_0 be an extremal point of $M^{(k)}$,

$$\|M^{(k)}\| = |M^{(k)}(z_0 - 0)| \qquad \text{or} \qquad \|M^{(k)}\| = |M^{(k)}(z_0 + 0)|.$$

We assume for concreteness that

$$\|M^{(k)}\| = |M^{(k)}(z_0 - 0)|.$$

There exists a point u such that

$$|M_0^{(k)}(z_0 + u)| = \max_x (M_0^{(k)}(x) \cdot \operatorname{sgn} M^{(k)}(z_0 - 0)). \tag{17}$$

The difference

$$s(x) = M_0(x+u) - c_k M(x), \qquad c_k = |M_0^{(k)}(z_0 + u)|/\|M^{(k)}\|,$$

has $2N$ sign changes on the period

$$\operatorname{sgn} s(z_i - u) = \operatorname{sgn} M_0(z_i) \qquad (i = 1 : 2N)$$

because in view of (16)

$$|M_0(z_i)| = \|M_0\| > \|M\| \cdot \|M_0^{(k)}\|/\|M^{(k)}\|$$
$$\geqslant |M(z_i - u)|\ |M_0^{(k)}(z_0+u)|/\|M^{(k)}\|.$$

Hence $\mu(s^{(k)}) \geqslant 2N$. From (17) we obtain

$$s^{(k)}(z_0) = 0$$

and

$$s^{(k+1)}(z_0) = 0 \qquad (\text{if } k < r-2).$$

Thus, for $k < r-2$

$$\mu(s^{(r-1)}) \geqslant 2N + 1. \tag{18}$$

On the other hand, in view of (15), $c_k \in (0, 1)$, inequality (18) contradicts (11).

Let $k = r-1$. In this case z_0 is the node of M, $z_0 + u$ is the node of M_0. The function $s^{(r-1)}$ can change sign at the point z_0 from "minus" to "plus" only. Hence, by the lemma $\mu(s^{(r-1)}) \leqslant 2N-1$. But $\mu(s) \geqslant 2N$ and $\mu(s^{(r-1)}) \geqslant 2N$.

Let $k = r-2$. If z_0 is a node of M, then $M^{(r-2)}(z_0) > 0$ because $M \in M_N^{r+}$ and $M^{(r-1)}(x)$ can change sign from "plus" to "minus." Hence, $z_0 + u$ is a node of M_0. In this case if the derivative $s^{(r-2)}$ changes sign at the point z_0 then $s^{(r-1)}$ does not change sign at this point. By the lemma $\mu(s^{(r-1)}) \leqslant 2N-1$. But on the other hand $\mu(s^{(r-1)}) \geqslant 2N$. If $s^{(r-2)}$ does not change sign at the point z_0 then $\nu(s^{(r-2)}) \geqslant 2N+1$ and $\mu(s^{(r-1)}) \geqslant 2N+1$. This inequality contradicts (11).

If z_0 does not coincide with the nodes of M then $M^{(r-2)}(z_0) < 0$. $M^{(r-1)}(z_0) = 0$ and $M_0^{(r-2)}(z_0+u) < 0$, $M^{(r-2)}(z_0+u) = 0$. Hence, $s^{(r-1)}(z_0) = 0$ and $s^{(r-2)}$ does not change sign at the point z_0. Thus, $\nu(s^{(r-2)}) \geqslant 2N+1$ and $\mu(s^{(r-1)}) \geqslant 2N+1$. This inequality contradicts (11). Theorem 2 is proved. ∎

3

In [3] it was proved that for every $M \in M_N^r(A, B)$ ($M \in M_N^r$) there exists a monospline $M_0 \in M_N^{r0}(A, B)$ ($M_0 \in M_N^{r0}$) such that $|M_0(x)| \leqslant |M(x)|$ for every $x \in [0, 1]$. From the proof of this inequality it follows that the monospline has nonnegative coefficients a_i in the representation (4) (or (6)) and the same sign as M: $M_0(x) \cdot M(x) \geqslant 0$. Thus the following theorem holds.

THEOREM 3. *For every* $M \in M_N^r(A, B)$ $(M \in M_N^r)$ *there exists a monospline* $M_0 \in M_N^{r+}(A, B)$ $(M_0 \in M_N^{r+})$ *such that*

$$|M_0(x)| \leqslant |M(x)|, \qquad M_0(x) \cdot M(x) \geqslant 0 \qquad \forall x.$$

Now we apply this theorem to the theory of the best quadrature formulas. We consider the following classes of functions having $r-1$ absolute continuous derivatives on $[0, 1]$

$$W^r(u) = \{f\colon \operatorname{meas} E(|f^{(r)}| > u) = 0\},$$
$$W_p^r(u, v) = \{f\colon \|uf_+^{(r)} + vf_-^{(r)}\|_p \leqslant 1\},$$
$$W_{p,q}^r(u, v) = \{f\colon \|uf_+^{(r)}\|_p + \|vf_-^{(r)}\|_p \leqslant 1\},$$

where u and v are fixed positive integrable functions on $[0, 1]$ such that $1/u$ and $1/v$ are integrable. In addition we define,

$$g_+(x) = \max(g(x); 0), \qquad g_-(x) = \max(-g(x); 0).$$

$\tilde{W}^r(u)$, $\tilde{W}_p^r(u, v)$, $\tilde{W}_{p,q}^r(u, v)$ are the corresponding classes of 1-periodic functions.

THEOREM 4. *Among all quadrature formulas,*

$$\int_0^1 w(t)\, f(t)\, dt = Q_N(f) + \sum_{k \in A} b_k f^{(k)}(0) + \sum_{m \in B} c_m f^{(m)}(1) + R(f), \quad (19)$$

where w *is a fixed integrable function,* $\operatorname{meas} E(w \leqslant 0) = 0$, A *and* B *are fixed subsets of* Z_r *(if* $A = \varnothing$ *or* $B = \varnothing$ *then the corresponding sum equals zero),*

$$Q_N(f) = \sum_{i=1}^{n} \sum_{j=0}^{r-1} a_{ij} f^{(j)}(x_i), \qquad \sum_{i=1}^{n} \sum_{j=0}^{r-1} \operatorname{sgn} |a_{ij}| \leqslant N,$$

$0 < x_1 < \cdots < x_n < 1$, *the best formula exists for the class* $W^r(u)$ $(W_p^r(u, v)$, $W_{p,q}^r(u, v))$ *and has the form*

$$\int_0^1 w(t)\, f(t)\, dt = \sum_{i=1}^{N} a_i f(x_i) + \sum_{k \in A} b_k f^{(k)}(0) + \sum_{m \in B} c_m f^{(m)}(1) + R(f),$$

and $a_i > 0$ $(i = 1 : N)$, $(-1)^{k + \alpha_k} b_k > 0$ $(k \in A)$, $(-1)^{\beta_m} c_m > 0$ $(m \in B)$ *where* $\alpha_k(\beta_k)$ *is the number of elements of* $A(B)$ *that are less than* k.

THEOREM 5. *Among all quadrature formulas on a periodic setting,*

$$\int_0^1 w(t)\, f(t)\, dt = Q_N(t) + R(f), \quad (20)$$

the best formula exists for the class $\tilde{W}^r(u)$, $(\tilde{W}^r_p(u, v)$, $\tilde{W}^r_{p,q}(u, v))$ *and has the form*

$$\int_0^1 w(t)\, f(t)\, dt = \sum_{i=1}^{N} a_i f(x_i) + R(f), \qquad \sum_{i=1}^{N} a_i = \int_0^1 w(t)\, dt,$$

$a_i > 0$ $(i = 1 : N)$.

Proof. Let us prove Theorem 4 for the class $W^r_p(u, v)$. The proofs of other results are similar.

It is known (see, e.g., [1, 4]) that the error R of the best quadrature formula has the representation

$$R(f) = ((-1)^r/(r-1)!) \int_0^1 f^{(r)}(t)\, M(t)\, dt, \tag{21}$$

where $M \in M^r_N(A^1, B^1)$, $A^1 = \{i: r-1-i \in Z_r \backslash A\}$, $B^1 = \{i: r-1-i \in Z_r \backslash B\}$. The theorem follows from the following equality:

$$\begin{aligned} R(W^r_p(u, v)) &:= \sup_{f \in W^r_p(u, v)} |R(f)| \\ &= (1/(r-1)!) \max(\|u^{-1}M_+ + v^{-1}M_-\|_{p'}; \\ &\quad \|v^{-1}M_+ + u^{-1}M_-\|_{p'}) =: \|M\|_{u,v,p'}/(r-1)!, \end{aligned} \tag{22}$$

$p' = p/(p-1)$ because in view of Theorem 3

$$\inf_{Q_N, b_k, c_m} (r-1)!\, R(W^r_p(u, v)) = \inf_{M \in M^r_N(A^1, B^1)} \|M\|_{u,v,p'} = \|\bar{M}\|_{u,v,p'},$$

where $\bar{M} \in M^{r+}_N(A^1, B^1)$.

Now we establish the equality (22). Starting from (21) we obtain

$$\begin{aligned} (r-1)!\, |R(f)| &= \left| \int_0^1 (f_+(t)\, M_+(t) + f_-(t)\, M_-(t))\, dt \right. \\ &\quad \left. - \int_0^1 (f_-(t)\, M_+(t) + f_+(t)\, M_-(t))\, dt \right| \\ &\leqslant \max \left\{ \int_0^1 (f_+ M_+ + f_- M_-)\, dt; \right. \\ &\quad \left. \int_0^1 (f_- M_+ + f_+ M_-)\, dt \right\}, \end{aligned}$$

$$\int_0^1 (f_+ M_+ + f_- M_-)\,dt \leqslant \int_0^1 (uf_+ + vf_-)(u^{-1}M_+ + v^{-1}M_-)\,dt$$
$$\leqslant \|u^{-1}M_+ + v^{-1}M_-\|_{p'},$$
$$\int_0^1 (f_- M_+ + f_+ M_-)\,dt \leqslant \int_0^1 (uf_+ + vf_-)(u^{-1}M_- + v^{-1}M_+)\,dt$$
$$\leqslant \|v^{-1}M_+ + u^{-1}M_-\|_{p'}.$$

Thus,

$$(r-1)!\, R(W_p^r(u, v)) \leqslant \|M\|_{u,v,p'}.$$

On the other hand, we have

$$R(f_1) = \|u^{-1}M_+ + v^{-1}M_-\|_{p'}, \qquad R(f_2) = \|v^{-1}M_+ + u^{-1}M_-\|_{p'},$$

where $f_1, f_2 \in W_p^r(u, v)$,

$$f_1^{(r)}(x) = (u^{-p'}(x)\, M_+^{p'-1}(x) - v^{-p'}(x)\, M_-^{p'-1}(x))/\|u^{-1}M_+ + v^{-1}M_-\|_{p^1}^{p'/p},$$
$$f_2^{(r)}(x) = (u^{-p'}(x)\, M_-^{p'-1}(x) - v^{-p'}(x)\, M_+^{p'-1}(x))/\|v^{-1}M_+ + u^{-1}M_-\|_{p}^{p'/p},$$

and

$$(r-1)!\, R(W_p^r(u, v)) \geqslant \max(R(f_1), R(f_2)) = \|M\|_{u,v,p'}.$$

For the class $W_{p,q}^r(u, v)$ we have the following expression for error R:

$$(r-1)!\, R(W_{p,q}^r(u, v) = \max(\|u^{-1}M_+\|_{p'}; \|u^{-1}M_-\|_{p'};$$
$$\|v^{-1}M_+\|_{q'}, \|v^{-1}M_-\|_{q'}$$
$$=: \|M\|_{u,v,p',q'} \qquad (p' = p/(p-1), q' = q/(q-1),$$
$$\inf_{Q_N, b_k, c_m} (r-1)!\, R(W_{p,q}^r(u, v)) = \inf_{M \in M_N^r(A^1, B^1)} \|M\|_{u,v,p',q'}$$
$$= \|M_0\|_{u,v,p',q'}, \tag{23}$$

where $M_0 \in M_N^{r,+}(A', B')$.

4

Let $u(x)$ and $v(x)$ be two positive continuous functions on $[0, 1]$. By the theorem on snakes for monosplines (see [3, 5]) there is a unique monospline $\bar{M} \in M_N^{r0}(A, B)$ and a positive constant c such that

$$-v(x) \leqslant c\bar{M}(x) \leqslant u(x) \tag{24}$$

and there exist $\alpha := 2N + r + 1 - |A| - |B|$ points

$$0 \leqslant z_1 < \cdots < z_\alpha \leqslant 1$$

at which

$$c\bar{M}(z_i) = -v(z_i)\ (i \text{ odd}), \qquad c\bar{M}(z_i) = u(z_i)\ (i \text{ even}). \tag{25}$$

For an arbitrary monospline $M \in M_N^{r0}(A, B)$ $(M \not\equiv \bar{M})$

$$\|M\|_{u,v} > \|\bar{M}\|_{u,v} = 1/c \qquad (\|f\|_{u,v} = \|u^{-1}f_+ + v^{-1}f_-\|_\infty). \tag{26}$$

Indeed, if there exists a monospline $M \in M_N^{r0}(A, B)$ such that

$$\|M\|_{u,v} \leqslant \|\bar{M}\|_{u,v},$$

then in view of (24) and (25) $v(x) \geqslant \alpha - 1$, $s = \bar{M} - M$. But s is spline of $r-1$ order with at most $2N$ nodes and with minimal defect and $s^{(i)}(0) = 0$ $(i \in A)$, $s^{(j)}(1) = 0$ $(j \in B)$. Hence, $v(s) \leqslant \alpha - 2$. This contradiction proves (26). Thus, in view of Theorem 3 we have proved the following result.

THEOREM 6. *Let u and v be two positive continuous functions on* $[0, 1]$. *There exists a unique monospline $\bar{M}$ with minimal (u, v)-norm in $M_N^r(A, B)$, $\bar{M} \in M_N^{r+}(A, B)$. The monospline $\bar{M}$ has minimal (u, v)-norm if and only if the function $u^{-1}\bar{M}_+ + v^{-1}\bar{M}_-$ has $2N + r + 1 - |A| - |B|$ alternation points on* $[0, 1]$.

In view of (22) and (23) the following theorem holds.

THEOREM 7. *Let u and v be two positive continuous functions on* $[0, 1]$. *Let M_1 be the monospline with minimal (u, v)-norm ((u, u)-norm) in $M_N^r(A^1, B^1)$ and M_2 be the monospline with minimal (v, u)-norm ((v, v)-norm). If $\|M_1\|_{u,v} = \|M_2\|_{v,u}$ ($\|M_1\|_{u,u} = \|M_2\|_{v,v}$) then there exist exactly two best quadrature formulas of the form* (19) *for the class $W_1^r(u, v)$ ($W_{1,1}^r(u, v)$). They are defined by the nodes and the coefficients of the monosplines M_1 and M_2 (see, e.g.,* [1, 4]). *If $\|M_1\|_{u,v} \neq \|M_2\|_{u,v}$ ($\|M_1\|_{u,u} \neq \|M_2\|_{v,v}$) then this formula is unique and defined by the monospline having bigger norm.*

In a similar way the following theorem can be proved.

THEOREM 8. *Let u and v be two positive* 1*-periodic continuous functions. There exists a unique monospline M_ξ with minimal (u, v)-norm in $M_N^r(\xi)$, $M_\xi \in M_N^{r+}(\xi)$, where $M_N^r(\xi)$ and $M_N^{r+}(\xi)$ are the sets of monosplines from M_N^r and M_N^{r+} which have one fixed node at the point ξ. The monospline M has minimal (u, v)-norm in $M_N^r(\xi)$ if and only if the function $u^{-1}M_+ + v^{-1}M_-$ has $2N$ alternation points on the period.*

THEOREM 9. *Let u and v be two positive 1-periodic continuous functions and ξ be a fixed point. M_1 denotes the monospline with minimal (u, v)-norm $((u, u)$-norm) in $M_N^r(\xi)$ and M_2 denotes the monospline with minimal (v, u)-norm (v, v-norm). If $\|M_1\|_{u,v} = \|M_2\|_{v,u}$ then there exist exactly two best quadrature formulas for the class $\tilde{W}_1^r(u, v)$ $(\tilde{W}_{1,1}^r(u, v))$ of form* (20) *with fixed node x_1 at the point ξ. They are defined by the nodes and the coefficients of the monosplines M_1 and M_2. If $\|M_1\|_{u,v} \neq \|M_2\|_{v,u}$ $(\|M_1\|_{u,u} \neq \|M_2\|_{v,v})$ then this formula is unique and is defined by the monospline having bigger norm.*

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